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ON SOLVING EXTERNAL BOUNDARY VALUE PROBLEMS OF THE THEORY OF ELASTICITY USING THE METHOD OF BOUNDARY INTEGRAL EQUATIONS*

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An algorithm is proposed for the method of boundary integral equations for the second external problem of the theory of elasticity with Poisson's ratio close to or equal 1/2 for the first external problem.

The method of boundary integral equations for basic problems of the theory of elasticity leads to singular integral equations of the second kind /1/. In the case of the second internal and external and, also, in the case of the first internal problem, the spectral properties of integral operators enable us to apply the method of successive approximations for finding a solution /2/. That method allows to lower the demand for memory volume, in a certain meaning to a minimum, as it is necessary to store the approximation on each step, i.e. the information amount equal generally to that required for specifying the boundary conditions.

Below, problems are considered for which the standard course (in the variant of the method of potential, i.e. replacing the unknown displacement by a potential) results in integral equations that are not equivalent to the input problem. The method proposed here is similar to that used in /3/ for solving the external Dirichlet problem for the Laplace equation (see also /4/).

1. Let $u(x) = (u_1, u_2, u_3)$ be the displacement vector of the elastic body D completely filling the infinite part of space $R^3 \ni x$ with the boundary closed by a smooth surface S.

Let us determine the differential operator

$$L_\sigma = \Delta + (1 - 2\sigma)^{-1} \operatorname{grad} \operatorname{div}$$

The first and second basic problems are defined as follows:

$$L_\sigma u = 0, \quad x \in D; \quad u(x) = f(x), \quad x \in S \quad (\text{problem } 1) \tag{1.1}$$

$$L_\sigma u = 0, \ x \in D; \ T_{n\sigma} u \overset{\Delta}{=} \frac{E}{2(1-\sigma)} \sum_{j=1}^{3} \left[\frac{\sigma}{1-2\sigma} \delta_{ij} \operatorname{div} u + \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_k}\right] n_j = f \quad (\text{problem } 2)$$

where σ and E are, respectively, the Poisson and Young moduli, and $n = (n_1, n_2, n_3)$ is the vector of normal to S.

The fundamental solution of operator L_σ is the matrix V with components

$$V_{ij} = \frac{3}{8\pi E(1-\sigma)} \left[\frac{3-4\sigma}{|x-y|} \delta_{ij} + \frac{(y_i - x_i)(y_j - x_j)}{|x-y|^3}\right] \tag{1.2}$$

The method of the potential consists of the following substitution of variables: for problem 1 the solution is sought in the form of double layer potential

$$u(x) = \int_S [T_{ny} V(x, y)]' \varphi(y) \, d_y S$$

for problem 2 it is sought in the form of simple layer potential

$$u(x) = \int_S V(x, y) \varphi(y) \, d_y S$$

With this substitution the equations in D are satisfied, since $LV\ x, y) = -2\delta(x, y) I$ and the boundary conditions yield singular integral equations in φ /1/ (for problem 1±)

$$\mp \varphi(x) + \int [T_{ny} V(x, y)]' \varphi(y) \, d_y S = f(x), \quad x \in S \tag{1.3}$$

*Prikl.Matem.Mekhan.,Vol.47,No.3,pp.462-468,1983 for problem 2±)

$$\pm\varphi(x)+\int_S T_{nx}V(x,y)\varphi(y)\,d_yS=f(x),\quad x\in S \tag{1.4}$$

Thus plus and minus sign denote here that the equations relate to the inner and outer problems, respectively, and subscripts x, y indicate that differentiation of the stress operator is carried out with respect to these variables.

The singular integral operators in (1.3) and (1.4) are conjugate in

$$L_2(S)=\left\{\varphi:(\varphi,\varphi)=\int_S\sum_{i=1}^{3}(\varphi^i(x))^2\,dS<\infty\right\}$$

where they are continuous.

Subsequently for definiteness and certain simplicity, the equations and related operators on S will be considered in $L_2(S)$. In the presence of corresponding smoothness of boundary parts the whole of the expounded is transferred into the Hölder-Lipschitz space, for which it is necessary to use certain theorems of /1/.

Equations (1.3) and (1.4) may be written in the operator form

$$\mp\varphi+T^*\varphi=f,\quad \pm\varphi+T\varphi=f$$

From /1/ we have: for some $\delta=\delta(S)>0$ we have $\Sigma(T)\ni -1$, $\Sigma(T)\setminus\{-1\}\subset[-1+\delta,\ 1-\delta]$ and $N(I+T)=\{\psi_i\}_1^6$ and -1 belongs to the region of Fredholm properties of operator T. Here and subsequently $\Sigma(\cdot)$ and $N(\cdot)$ are, respectively, the spectral set and the space of zeros of operator in parentheses; ψ_i are linearly independent vectors of rigid displacement $a+[b\times x]$ (a and b are vector constants). Subsequently, we shall denote by $^{\perp}A$ the space of functions orthogonal in $L_2(S)$ to A: $^{\perp}A=\{\varphi: (\varphi,\psi)=0\ \ \forall\psi\in A\}$.

2. The singularity in the Lame equation (1.1) as $\varepsilon=1-2\sigma\to 0$ produces difficulties in numerical calculations for media with σ close to or equal $^1/_2$. To overcome this difficulty in problems $1^+, 2^+$ is not difficult (see /5/, where the existence of the limit of solution of problem 2^- as $\varepsilon\to 0$ was also proved). In problem 2^- the respective integral equations when $\varepsilon=0$ are equations on the spectrum, which hinders the seeking of solution in the form of simple layer potential and with small ε.

Let L_0, T_{n0} be operators acting on the pair (u, p) in the following manner:

$$L_0(u;p)=\left\{\frac{E}{3}\Delta u-\operatorname{grad}p;\ \operatorname{div}u\right\}$$

$$T_{n0}(u;p)=\frac{E}{3}\sum_{j=1}^{3}\left[\delta_{ij}p+\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right]n_j$$

Consider the problem

$$L_0(u;\ p)=0,\ x\in D^-;\ T_{n0}(u;\ p)=f,\quad x\in S \tag{2.1}$$

where D^- is the unbounded part of space.

Problem (2.1) was considered in the investigation of steady Stokes flow of viscous incompressible fluid /6/, although it was not of independent interest there. There u had the meaning of velocity, p of pressure, and $E/3$ was the coefficient of dynamic viscosity.

Let V_0, P be the fundamental solution of operator L_0: $L_0(V_0;P)=(-2\delta(x,y)I;\ 0)$, where V_0 is the matrix (1.2), for $\sigma={}^1/_2$ we have $P=(4\pi)^{-1}\operatorname{grad}|x-y|^{-1}$.

The equalities

$$V_\varepsilon=V_0+\frac{\varepsilon}{1+\varepsilon}V_1,\quad \operatorname{div}V_1=2P \tag{2.2}$$

$$T_\varepsilon=T_0+\frac{\varepsilon}{1+\varepsilon}T_1 \tag{2.3}$$

are valid, and T_0 is absolutely continuous and T_1 is a continuous operators. Here and subsequently the subscript ε denotes a quantity that is determined for $\sigma=(1-\varepsilon)/2$.

Moreover (see /6/)

$$-1,1\in\Sigma(T_0),\quad \exists\delta>0:\ \Sigma(T_0)\setminus\{-1,1\}\subset[-1+\delta,\ 1-\delta]$$
$$N(-I+T_0)=\{n\},\quad N(I+T_\varepsilon^*)=\{\psi_\varepsilon\}_{i=1}^6$$

The substitution

$$u(x)=\int_S V_0(x,y)\varphi(y)\,d_yS,\quad p(x)=\int_S\sum_{k=1}^{3}P^k(x,y)\varphi^k(y)\,d_yS$$

reduces problem (2.1) to the Fredholm equation in φ

$$-\varphi + T_0\varphi = f \tag{2.4}$$

which is solvable only under condition $f \in {}^{\perp}N(-I + T_0^*)$, and, consequently, the solution of problem (2.1) cannot be represented in the form of simple layer potential.

The presence of point $1 + O(\varepsilon)$ in sepctrum T_ε makes difficult the solution of Eq.(1.4) that corresponds to problem 2^- at small ε.

Prior to giving the algorithm whose stability is independent of ε, we shall make a few remarks.

1°. If λ is a pole of order m of the resolvent of operator T, then the space H where T is acting can be represented in the form of a direct sum /7/

$$H = N((-\lambda I + T)^m) \oplus {}^{\perp}N((-\lambda I + T^*)^m) \tag{2.5}$$

2°. Let $\Sigma_n(T_\varepsilon)$ be the spectrum of T_ε contraction on ${}^{\perp}N(I + T_\varepsilon^*)$, then $\Sigma_N(T_\varepsilon) = \Sigma(T_\varepsilon) \setminus \{-1\}$, which can be verified directly.

3°. Let $T_{\varepsilon 1} = T_\varepsilon - (n, \cdot)\, n$. Then $\Sigma(T_{01}) \subset \Sigma(T_0) \setminus 1$, and since $n \in {}^{\perp}N(I + T_\varepsilon^*)$, we have

$$\Sigma_N(T_{01}) \subset [-1 + \delta,\ 1 - \delta],\ \delta > 0 \tag{2.6}$$

Indeed, let $\lambda \in \Sigma(T_{01})$. Then there exist a φ such that $T_{01}\varphi = \lambda\varphi$. According to 1° $\varphi = c \cdot n + \overline{\varphi}$, $\overline{\varphi} \in {}^{\perp}N(-I + T_0^*)$ and $T_{01}\varphi = \lambda\, cn + \lambda\overline{\varphi}$. Since the expansion of (2.5) is unique and $T_0{}^{\perp}N(-I + T_0^*) \subset {}^{\perp}N(-I + T_0^*)$, hence $T_0\overline{\varphi} = \lambda\overline{\varphi}$. Therefore either $|\lambda| < 1$, or $\overline{\varphi} = \text{cosnt}\cdot n$ and $T_{01}\varphi = 0$.

4°. From (2.3) and (2.6) follows the existence of $\varepsilon_0 > 0$ and independent from ε $\delta_0 > 0$ such that for $\varepsilon \in [0, \varepsilon_0]$

$$\Sigma(T_{\varepsilon 1}) \subset [-1 + \delta_0,\ 1 - \delta_0],\ \| T_{\varepsilon 1} \|_N = q < 1 \tag{2.7}$$

5°. Since $(I + T_\varepsilon) L_2(S) \subset {}^{\perp}N(I + T_\varepsilon^*)$, it follows from (2.7) that the operator

$$R_\varepsilon = -\frac{1}{2}\left[I + \sum_{k=0}^{\infty} T_{\varepsilon 1}^k (I + T_\varepsilon)\right]$$

exists and is continuous. Moreover

$$\| R_\varepsilon - R_0 \| = O(\varepsilon) \tag{2.8}$$

We represent solution of problem 2^- in the form (a particular solution for $v(x)$ in another form was used in /8/)

$$u_\varepsilon(x) = \int_S V_\varepsilon(x, y)\, \varphi_\varepsilon(y)\, d_yS + c_\varepsilon v(x);\ v(x) = \mathbf{grad}\, |x_0 - x|^{-1} \tag{2.9}$$

where $x_0 \in R^3 \setminus (D \cup S)$ is a fixed point. Since for any function $F \in C^3(D)$ the equality

$$L\,\mathrm{grad}\, F = \frac{2(1-\sigma)}{1-2\sigma}\,\mathrm{grad}\,\Delta F$$

is valid, we have $Lu_\varepsilon = 0$, $x \in D^-$.

We have on S the equation in φ_ε, c_ε

$$-\varphi_\varepsilon + T_\varepsilon\varphi_\varepsilon = f - c_\varepsilon T_{n\varepsilon} v \tag{2.10}$$

We assume $c_\varepsilon = (n, R_\varepsilon f)/(n, R_\varepsilon, T_{n\varepsilon} v)$ (that the denominator is nonzero is proved below).

The solution of (2.10) is

$$\varphi_\varepsilon = R_\varepsilon f - [(n, R_\varepsilon f)/(n, R_\varepsilon T_{n\varepsilon} v)]\, R_\varepsilon T_{n\varepsilon} v$$

Indeed, φ_ε is the solution of Eq.(2.10) obtained by the substitution $T_\varepsilon \to T_{\varepsilon 1}$ and the statement follows from the equality $(\varphi_\varepsilon, n) = 0$.

Thus u_ε from (2.9) is the solution of problem 2^-, and the solution of (2.1) is the pair (u_0, p)

$$u_0(x) = \int_S V_0(x, y)\, \varphi_0(y)\, d_yS,\quad p(x) = \int_S \sum_{k=1}^{3} P^k(x, y)\, \varphi_0{}^k(y)\, d_yS \tag{2.11}$$

$$\varphi_0 = R_0 f - [(n, R_0 f)/(n, R_0 T_{n0} v)]\, R_0 T_{n0} v$$

$$\left(T_{n0} v \stackrel{\Delta}{=} \lim T_{n\varepsilon} v = T_{n0}(v; 0) = \left\{\frac{E}{3} \sum_{j=1}^{3} \frac{\partial^2}{\partial x_i \partial x_j} |x_0 - x|^{-1} \cdot n_j\right\}_{i=1}^{3}\right)$$

which is valid by virtue of

$$\operatorname{div} v = \Delta (|x_0 - x|^{-1}) = 0, \quad x \in D^- \cup S$$

By virtue of (2.2), (2.3) and (2.8) we have $\| \varphi_\varepsilon - \varphi_0 \| = O(\varepsilon)$, $\| u_\varepsilon - u_0 \| = O(\varepsilon)$.

It is essential that the action of R_ε on the function can be calculated using the method of successive approximations that converges not slower than the geometric progression with an exponent independent of ε.

It remains to show that for ε close to or equal zero we have $(n, R_\varepsilon T_{n\varepsilon} v) \neq 0$.

Since $(n, R_\varepsilon T_{n\varepsilon} v) = (n, R_0 T_{n0} v) + O(\varepsilon)$, it is sufficient to show that $n, R_0 T_{n0} v) \neq 0$. Let us assume to contrary, then $R_0 T_{n0} v$ is the solution of equation

$$-\varphi_0 + T_0 \varphi_0 = T_{n0} v$$

From this follows that the pair $(v; 0)$, as the solution of problem (2.1) with boundary conditions $f = T_{n0} v$ can be represented by the simple layer potential of density $\varphi_0 + \text{const} \cdot n$. Since

$$\int_S V_0(x, y)\, n(y)\, d_y S = 0, \quad \int_S \sum_{k=1}^{3} P^k(x, y)\, n_k(y)\, d_y S = 0, \quad x \in D^-$$

we have $u_0(x) = v(x), p(x) = 0, x \in D^-$, where $u_0(x), p(x)$ are defined by formulas (2.11)

The last equality means that $\varphi_0 = c \cdot n$, which is impossible, since $T_{n0} v \not\equiv 0$, $x \in S$.

Let us briefly consider the case when S contain several components of connectedness. Let, for example, $S = S_1 \cup S_2$, S_i be a closed smooth surfaces and $S_1 \cap S_2 = \varnothing$, D_i^+ $(i = 1, 2)$ be the bounded part of R^3 with boundary $\partial D_i^+ = S_i$, $D_i^- = R^3 \setminus (D_i^+ \cup S_i)$. Let an elastic body occupy the region $D = D_2^- \cap D_1^+ \neq 0$, and assume that the normal n is directed outward of D.

Let us consider the problem

$$L_\varepsilon u_\varepsilon = 0, \quad x \in D; \quad T_{n\varepsilon} u_\varepsilon = f, \quad x \in S$$

Here (and similarly below) $f = \{f^1; f^2\}$, where f^i are vector functions defined on S_i.
We seek a solution of the problem

$$u_\varepsilon(x) = \sum_{i=1}^{2} \int_{S_i} V(x, y)\, \varphi_\varepsilon^i(y)\, d_y S + c_\varepsilon \operatorname{grad} |x_0 - x|^{-1}, \quad x_0 \overline{\in} D \cup S$$

Let us determine the matrix operator

$$T_\varepsilon = \{T_{\varepsilon ij}\}, \quad T_{\varepsilon ij} \varphi = \int_{S_j} T_{n\varepsilon} V_\varepsilon(x, y)\, \varphi^j(x, y)\, d_y S$$
$$x \in S_i; \; i, j = 1, 2$$

For $\varphi_\varepsilon = \{\varphi_\varepsilon^1; \varphi_\varepsilon^2\}$ we have the equation

$$\varphi_\varepsilon + T_\varepsilon \varphi_\varepsilon = f - c_\varepsilon T_{n\varepsilon} \operatorname{grad} |x_0 - x|^{-1}$$

The following formulas can be verified:

$$T_\varepsilon = T_0 + \varepsilon T_1, \; \| T_1 \| < c \text{ (is independent of } \varepsilon)$$
$$N(I + T_0)_N = \{0; n\}; \; N(-I + T_\varepsilon^*) = \{0; \psi_i^1\}$$
$$(-I + T_\varepsilon) L_2(S) \subset {}^{\perp} N(-I + T_\varepsilon^*)$$

If we denote by $T_{\varepsilon 1}$ the operator obtained from T_ε by the substitution $T_{22} \to T_{22} - n^2 (n^2, \cdot) L_2(S_2)$, we find that there exists $\varepsilon_0 > 0$, such that for $\forall \varepsilon \in [0, \varepsilon_0]$ exists the bounded operator

$$R_\varepsilon = \frac{1}{2} \left[I + \sum_{k=0}^{\infty} (-T_\varepsilon)^k (-I + T_\varepsilon) \right]$$

Further constructions repeat in essence those effected in the case of a simply connected surface.

3. Let S be a closed smooth simply connected surface which divide R^3 in two regions: the finite D^+ and infinite D^-.

The difficulty of solving problem 1^- using the method of integral equations is in that the solution is not representable in the form of a double layer potential. It was proposed in /1/ to seek the solution in the form of a sum of the simple and double layers potentials of the same density. In the computing plan the more convenient is the following method.

We write

$$u(x) = \int_S [T_{ny} V(x, y)]' \varphi(y)\, d_y S + \sum_{i=1}^{6} c_i \int_S V(x, y)\, \psi_i(y)\, d_y S$$

where c_i are unknown constants and ψ_i are, as before, the vectors of rigid displacement.

For φ, c_i we have the equation

$$\varphi + T^*\varphi = F \overset{\Delta}{=} f - \sum_{i=1}^{6} V * \psi_i \cdot c_i \qquad (3.1)$$

$$\left(V * \psi_i = \int_S V(x, y)\, \psi_i(y)\, d_y S\right)$$

Let $\{\varphi_i(x)\}_{i=1}^{6} = N(I + T)$. For the solvability of (3.1) it is necessary and sufficient that conditions

$$(F, \varphi_i) = 0, \quad i = 1, 2, \ldots, 6 \qquad (3.2)$$

are satisfied. These conditions determine c_i uniquely. Indeed, since $V * \varphi_i = \psi_i$, and functions $V(x, y)$ is symmetric, we have

$$(F, \varphi_i) = (f, \varphi_i) - \sum_{j=1}^{6} c_j (\psi_i, \psi_j)$$

Thus φ, c_i that satisfies (3.1) uniquely defines the solution, since by virtue of the generalized Gauss theorem

$$\int_S [T_{ny} V(x, y)]\, \psi_i(y)\, d_y S = 0, \quad x \in D^-$$

Let us pass from (3.1) to the equation

$$\varphi + T_1^*\varphi = F \quad (T_1^* = T^* + \Sigma \psi_i(\cdot, \psi_i)) \qquad (3.3)$$

The following statement is correct (see proof below):

$$\Sigma(T_1) \subset \Sigma(T)\{-1\}$$

The conditions (3.2) and $(\varphi, \psi_k) = 0$ $(k = 1, \ldots, 5)$ are equivalent, hence, when one of them is satisfied the solution of (3.3) is also the solution of (3.1). This implies that solution of (3.3) can be obtained using the Neumann series, since

$$R \overset{\Delta}{=} (I + T_1^*)^{-1} = \sum_{k=0}^{\infty} (-T_1^*)^k$$

Proof. Point -1 is a simple pole of the resolvent of operator T^* /1/. We shall show in the beginning that $\Sigma(T_1^*) \subset \Sigma(T) \setminus \{-1\}$. Since the regions of Fredholm properties of operators T_1^* and T are the same, it is sufficient to consider only the point spectrum.

Let λ, φ be, respectively, the eigenvalue and the eigenfunction of operator T_1^*. We multiply the equation $\lambda\varphi - T_1^*\varphi = 0$ by ψ_k. We have $\lambda(\varphi, \psi_k) = 0$ hence when $\lambda \neq 0$ we have $(\varphi, \psi_k) = 0$ $(k = 1, 2, \ldots, 6)$ and $\Sigma(T_1^*) \subset \Sigma(T^*)$. Let $-1 \in \Sigma(T_1^*)$, then from $\varphi^1 + T_1^*\varphi^1 = 0$, as proved, we have $(\varphi^1, \psi_k) = 0$ and $\varphi^1 + T^*\varphi^1 = 0$. By the Fredholm theorem from this follows in turn the existence of functions φ^2 such that $\varphi^2 + T^*\varphi^2 = \varphi^1$, which contradicts the condition of simplicity of pole 1, since in that case φ^2 is an adjoint function.

We multiply now (3.1) by φ_k $(\{\varphi_k\}_1^6 = N(I + T))$ and obtain

$$(F, \varphi_k) = \sum_{i=1}^{6} (\varphi, \psi_i)(\psi_i, \psi_k)$$

By virtue of simplicity of pole 1 vectors ψ_i, φ_k it can be biorthonormalized $(\psi_i, \varphi_k) = \delta_{ik}$. From this follows the statement of second part.

The algorithm of solution of problem 1^- comes in the following two stages:

1) the calculation of Rf, $R(V * \varphi_i)$,

2) determination of c_i from the solution of six linear algebraic equations

$$\sum_{i=1}^{6} c_i (RV * \psi_i, \psi_j) = -(Rf, \psi_j)$$

The extension to the case when S consists of several simply connected components is elementary.

The method presented here may also be applied in the case when on some surfaces bounding the body stresses are specified.

Let, for example, the body lie between the surfaces S_0 and S_1 with S_1 contained inside S_0 and $S_0 \cap S_1 = \varnothing$. Repeating the reasoning in /9/ presented there for a similar problem in the case of Laplace equations, we obtain a system of equations with the same spectral properties. Thus, if on S_1 are given displacements and S_0 stresses, we obtain an equation at an isolated point of the spectrum, with the remaining part of the spectrum lying inside the interval $(-1, 1)$. The eigenfunctions are zero on S_0 and rigid displacements on S_1. Further investigations virtually repeat the foregoing.

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